

A fourth 'I' of poverty ?

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Abstract Current poverty measurement methodology does not allow a definitive analysis of changes in distribution, through time or between countries, which involve changes in the number or proportion of poor people. By re-opening some of the discussion which has taken place around the incidence, intensity and inequality aspects of poverty, and by revisiting the continuity and transfer axioms, we show that the Bourguignon and Fields poverty index allows considerable ethical flexibility when its parameters are used to full advantage. Significantly, a fourth dimension of poverty, the injustice of it, corresponding closely with Rawls's concern for the least advantaged, can also be admitted into the picture once the poverty aversion parameter in the Bourguignon and Fields index is fully understood and used appropriately. A novel application leads to a perspective upon the entire class of relative poverty indices which has not been seen before, and also generates both potentially interesting new poverty indices and wider scope for cogent measurement.

1. Introduction

Different poverty indices can exhibit startlingly different behaviours in response to the simplest of distributional changes. How can this be? In this paper, we re-examine a number of assumptions that have become conventional in poverty analysis, relating to continuity at the poverty line and what happens (or should happen) when people cross this line. Our findings provide illumination for such different behaviours, and, we hope, will give encouragement as well as added flexibility to analysts coping with real-world measurements where the number and/or proportion of poor people differs between distributions being compared. We frame our discussion around Sen's "three 'I's of poverty" which have become highly influential - namely the *incidence*, *intensity* and *inequality* dimensions of aggregate poverty. From this reconsideration also emerges a possible "fourth 'I' of poverty", hence the title of our paper.

Sen's (1976) development of his poverty index is motivated by the need for a poverty index to be adequately informative on the situation of the poor. Sen values the informative content inherent in the headcount ratio (H) and in the income-gap ratio (I), and asserts that "Both should have some role in the index of poverty" (p. 223). The proportion of poor individuals in a society provides information

about the incidence of poverty. The extent to which poor incomes fall short from the poverty line gives indications about the intensity of poverty. However, except in the unlikely case of a perfectly egalitarian income distribution below the poverty line - Axiom N in Sen (1976) - the use of H and I alone is challenged on the ground of their “crudeness”. The blame is on the silence about how incomes - or, equivalently, poverty shortfalls - are distributed among the poor. Such considerations motivated Sen to develop a “composite measure P” able to “take note of the inequality among the poor” where “G [the Gini coefficient of the poor] provides this information” (p. 227).

Various distribution-sensitive indices have been subsequently proposed in the literature, replacing the rank-order weighting used by Sen with other ways to take into account inequality below the poverty line. For example, the well-known parametric P_α class (Foster *et al.*, 1984, henceforth FGT) when $\alpha = 2$ adopts C_p^2 - the squared coefficient of variation of poor incomes - while for $\alpha = 0$ and $\alpha = 1$ it corresponds to H and HI respectively. Thanks to the incorporation of C_p^2 , $P_{\alpha=2}$ “indeed may be expressed as a combination of this inequality measure, the headcount ratio and the income-gap ratio in a fashion similar to Sen (1976)” (p. 761).¹

From the above, it should come as no surprise that only distributional-sensitive indices are affected by disequalizing distributional changes which leave the incidence and intensity of poverty unaltered - for example, with H and I staying the same, the Sen index and $P_{\alpha=2}$ increase. But if we wonder which behaviour we should expect from poverty indices that are informative *on all three 'I's*, there is definitely something more interesting to say. Two closely interrelated lines of investigation are opened, which we pursue in this work.

On the one hand, how is it possible that different composite measures respond in different ways to distributional changes affecting the ‘I’s of poverty? For example, consider an income distribution among 6 persons, $\mathbf{y} = (\$4, \$7, \$8, \$9, \$20, \$30)$, and a poverty line of $Z = \$10$. Now the poorest person gives \$1 to the one with \$9, and the incomes become $\mathbf{y}' = (\$3, \$7, \$8, \$10, \$20, \$30)$. As an outcome of that regressive transfer, the Sen index falls while $P_{\alpha=2}$ increases. We shall show how the Bourguignon and Fields (1997), henceforth BF, class of indices can be a tool allowing the accommodation of a rich

¹ Where $C_p^2 = \sum_{i=1}^q \frac{(\bar{y}_p - y_i)^2}{q\bar{y}_p^2}$, FGT show that $P_{\alpha=2}$ can be written as $H \left[I^2 + (1-I)^2 C_p^2 \right]$. Sen’s (1976) index takes the form $H \left[I + (1-I) G_p \right]$ where G_p is the Gini coefficient of poor incomes.

array of value judgements in response to the question in Kundu and Smith (1983): “How should poverty indices behave with respect to transfers which alter the size of the poor population?” (p. 430).

On the other hand, the need for a deeper understanding of the real meaning of the three ‘I’s as well as the need for them to be complemented with a fourth ‘I’ emerges clearly from the following simple numerical example. Consider a transfer in which the second poorest person in distribution $\mathbf{y} = (\$4, \$7, \$8, \$9, \$20, \$30)$ gives \$1 to the one with \$8, and the incomes become $\mathbf{y}'' = (\$4, \$6, \$9, \$9, \$20, \$30)$. In the new distribution, the incidence and intensity of poverty are the same whilst the inequality has increased. It looks natural that the composite measure $P_{\alpha=2}$ increases. Suppose now that from distribution \mathbf{y} the second poorest gives \$2 to the one with \$8 and \$1 to the one with \$9, and the incomes become $\mathbf{y}''' = (\$4, \$4, \$10, \$10, \$20, \$30)$. Still, the intensity of poverty has remained the same but both the incidence of poverty and the inequality among the poor have evidently decreased. How come that $P_{\alpha=2}$ tells us that poverty has increased? Moreover, $P_{\alpha \rightarrow \infty}$ would rank equally distributions \mathbf{y} and \mathbf{y}''' . Given that for α increasing indefinitely the customary three ‘I’s become virtually irrelevant, which is the ‘I’ informing such a comparison?

The structure of the paper is as follows. In Section 2, we review the value judgements which underpin the axioms for continuity, discontinuity and regressive transfers which are used in the poverty literature. In Section 3, we examine what happens when someone crosses the poverty line as the result of a regressive transfer in the case of the BF index. Minor adaptations provide the additional degree of ethical flexibility we have spoken of. In Section 4, we make the case for a fourth ‘I’ of poverty; also, by carefully interpreting the relevance of the ‘inequality’ of poverty, we are led to draw a parallel with the approach known as ‘prioritarianism’ in social justice theory. In Sections 5 and 6, the analytical tools developed earlier in the paper are jointly employed for a deeper understanding of the workings not only of the FGT and BF classes but also of all relative poverty indices, as to their distributional properties, their encapsulation of the four ‘I’s of poverty and their prioritarian stance; a potentially interesting new class of indices comes as a natural outcome. Section 7 concludes.

2. Value judgements: crossings of the poverty threshold

The attractiveness of a ‘smooth’ poverty function rests in the idea that “given a very small change in a poor person’s income, we could not expect a huge jump in the poverty level” (Zheng, 1997, p. 131). But should that hold also at the poverty line? While for Watts (1968) “poverty is not really a discrete condition. One does not immediately acquire or shed the afflictions we associate with the notion of

poverty by crossing any particular poverty line” (p. 325), Donaldson and Weymark (1986) do argue that the practical difficulties in measuring income make continuity a reasonable requirement, but acknowledge that “..on the other hand, the use of a poverty line to sharply demarcate the rich from the poor suggests, but does not require, that a poverty index might be discontinuous at the poverty line” (p. 674).

BF see two distinct aspects to the social welfare losses due to lack of adequate income. One arises simply because people are poor, in the sense that their income level does not allow them to fulfil the “accepted conventions of minimum needs” (Sen, 1979: 291). The other reflects the consideration that poverty becomes harsher the further the individual’s income falls below the poverty line. Therefore, BF suggest that *i*), a desirable function should take into account the continuous aspect of the welfare loss from poverty - they choose to focus upon the FGT class but hint at other distribution sensitive measures for this – and *ii*), for each poor individual, a constant δ should be added by virtue of his condition of being poor. In their own words, “a ‘fixed loss’ from poverty ... arises in addition to the income-dependent ‘variable loss’ from poverty” (p. 158). However BF make no recommendation about the precise magnitude of the fixed loss - i.e. the parameter δ - which “must be set by the observer” (p. 158). If the absoluteness of poverty in capabilities space (Sen, 1982) is accepted, then something highly significant indeed happens when an income unit crosses the poverty line. The last penny given to a poor person – the one which lifts him out of poverty – must then have a disproportional effect compared to the other pennies given. Continuity at the poverty line would make it impossible to accommodate this value judgement. Indeed, then, continuous indices are built upon the rejection of the notion of absoluteness of poverty in capabilities space, and upon the belief that reaching the poverty line does not provide any ‘highly significant’ social welfare gain.²

As a consequence, continuity ensures that a distribution-sensitive index increases as the result of *any* regressive transfer, even though the recipient may be taken out of poverty. We will name such a requirement the *unrestricted transfer axiom*, which for the ease of expression will be denoted by **UTA**; **RTA** will instead be used to refer to the *restricted transfer axiom*, according to which only those

² Lewis and Ulph (1988) argue for a jump discontinuity in utilitarian welfare at the poverty line, in a microeconomic model with an indivisible participation good, one unit of which, if affordable, takes away the shame of poverty – and causes a jump in indirect utility. For these authors, welfare *à la* Atkinson (1970) becomes $W_F = U(\mu_F) - [U(\mu_F) - \int_0^\infty U(x)f(x)dx] - g.H(F|Z)$ in which there are subtractions from the pure size measure $U(\mu_F)$ for the costs of inequality and poverty respectively. For Lewis and Ulph, in fact, “none of the writers who have tried to incorporate the distribution of income amongst the poor into their measure of poverty have adequately explained why this feature of income distribution should matter when measuring poverty (as distinct from inequality)” (*ibid.*, page 119).

regressive transfers in which the recipient remains poor necessarily increase poverty. There is a fundamental difference in the intuition behind these two. The principle informing **UTA** is immediate and appealing: a transfer from a poor individual to a less poor one is *always* a bad thing. The satisfaction of **RTA** instead suggests the idea that “a reduction of the number of the poor might under certain circumstances compensate a rise in the extent of penury of those who remain below the poverty line” (Sen, 1982: 33). In order to complete the picture, we introduce an alternative property, which we call the ‘rescue axiom’, denoted by **RA**, which calls for a decrease in poverty as the result of any regressive transfer lifting the recipient out of poverty. To the advocates of **RA**, for whom the only tool currently available in the literature is the ‘crude’ headcount, crossing the poverty threshold is *always* a good thing.

Whether a decrease in the number of poor individuals at such a cost should be welcomed or not is surely a troublesome issue. Discussing anti-poverty budgetary exercises, Subramanian (1997) throws a parallelism with the ‘lifeboat dilemma’ proposed by utilitarian philosophers which would well suit a discussion in which poverty is seen as a welfare loss from inadequate income. The issue under study is a good candidate for membership of the well-known “class of human problems which can be called ‘no technical solution problems’” (Hardin, 1968: 1243).

Our business is not to look for an ultimate solution where there is none, but to provide technical solutions accommodating different plausible value judgements. We investigate comprehensively the possibilities for a poverty index to respond to crossings of the poverty line as the result of regressive transfers, and we do it in tandem with the analysis of the case for a poverty index to be continuous or jump-discontinuous at the poverty line. The FGT class P_α and the BF class $P_{\alpha,\delta}$ represent an ideal field of investigation for the analytics of these issues, being respectively continuous and jump-discontinuous at the poverty line. As will be seen, the appropriate choice of the parameter δ in the BF class allows the accommodation of **UTA**, **RTA** or **RA** within a composite measure, which is monotonic and convex in the poor sub-domain, and which does not suffer from any ‘crudeness’. Clear-cut behaviours in response to regressive transfers lifting an individual out of poverty are hence warranted.

3. The BF index and crossings of the poverty threshold

Consider a fixed and finite set of individuals $A = \{1, 2, \dots, n\}$ and let $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}_n^{>0}$ be the corresponding vector of incomes arranged in non-decreasing order, where y_i is the income of the i^{th} individual. Take an exogenous poverty line $z \in \mathbb{R}^{>0}$ and define the individuals in the subset

$Q = \{1, 2, \dots, q\} \subseteq N$ with $|Q| = q$ as poor. We adopt the weak definition of the poor: y_q is the largest income *smaller* than z .³ Let a function $P(y; z) : \mathbb{R}_n^{>0} \times \mathbb{R}^{>0} \rightarrow \mathbb{R}^{\geq 0}$ evaluate aggregate poverty as a normalized sum of deprivation values. The FGT and the BF class of poverty measures are, respectively:

$$(1) \quad P_\alpha(y; z) = \frac{1}{n} \sum_{i=1}^n P_{\alpha,i}, \text{ with } P_{\alpha,i}(y; z) = \begin{cases} (\Gamma_i)^\alpha & \text{if } 0 < y_i < z \\ 0 & \text{if } y_i \geq z, \end{cases}$$

where $\alpha \in \mathbb{R}^{\geq 0}$ is interpreted by the authors as a parameter of poverty aversion, and

$$(2) \quad P_{\alpha,\delta}(y; z) = \frac{1}{n} \sum_{i=1}^n P_{\alpha,\delta,i}, \text{ with } P_{\alpha,\delta,i}(y; z) = \begin{cases} \delta + (\Gamma_i)^\alpha & \text{if } 0 < y_i < z \\ 0 & \text{if } y_i \geq z, \end{cases}$$

where $\Gamma_i = \frac{z - y_i}{z}$, $\delta \in \mathbb{R}^{>0}$ and α receives the same interpretation as in $P_{\alpha,i}$ but is taken to exceed unity in order to enjoy the larger set of properties associated to strict convexity below z . Clearly we have:

$$(3) \quad P_{\alpha,\delta} = \frac{1}{n} \sum_{i=1}^q [\delta + P_{\alpha,i}] = \delta H + P_\alpha.$$

For a regressive transfer, call the increase in the welfare loss from poverty of the donor Δ_D^+ and the decrease in the welfare loss from poverty of the recipient Δ_R^- . We normally consider $\Delta_D^+ > |\Delta_R^-|$: the net effect of the regressive transfer is an increased welfare loss from poverty. In the case of a line crossing, $|\Delta_R^-|$ may be thought larger than Δ_D^+ , entailing a gain overall rather than a loss. Given convexity, the satisfaction of the inequality $\Delta_D^+ < |\Delta_R^-|$ is not of interest when the recipient remains poor.

³ In BF's discrete model, there are no ties and nobody is located at the poverty line ($x_i < x_{i+1} \forall i$ and $x_Q < Z < x_{Q+1}$). In their model with continuously distributed incomes, positive density is allowed in a neighbourhood of the poverty line. FGT admit the possibility of people at the poverty line, and count such people as poor even though their presence has no impact on their index when $\alpha > 0$ (since $\lim_{\Gamma \rightarrow 0} \Gamma^\alpha = 0$ for $\alpha > 0$). When $\alpha = 0$, FGT say that their index reduces to the headcount ratio, but

since $\lim_{\Gamma \searrow 0} \Gamma^0 = 1$ and 0^0 is indeterminate, there is a problem with this.

Distribution-sensitive measures depict poverty as getting harsher *at increasing rates* the further we get below the poverty line - assuming differentiability, $\frac{\partial^2 P}{\partial y_i^2} > 0$ in the “poor” domain. As a consequence, for a regressive transfer the inequality $\Delta_D^+ > |\Delta_R^-|$ is always verified for continuous distribution-sensitive indices such as $P_{\alpha>1}$. The reason why such indices always increase is that they only deal with variable losses from poverty, while, conversely, $P_{\alpha=0}$ remains unchanged because it instead deals only with the fixed loss.

Jump-discontinuous distribution-sensitive indices such as $P_{\alpha,\delta}$ always increase if the recipient remains poor - thus satisfying **RTA**. In such cases the ‘fixed’ loss from poverty is not affected. However, once we consider transfers that take the recipient over the poverty threshold, $|\Delta_R^-|$ can be written as $\delta + |\tilde{\Delta}_R^-|$, where $\tilde{\Delta}_R^-$ represents the variable component of the variation. The condition $\frac{\partial^2 P}{\partial y_i^2} > 0$ ensures that $\Delta_D^+ - |\tilde{\Delta}_R^-| = \tilde{\Delta} > 0$ but is silent on the sign of $\Delta = \tilde{\Delta} - \delta$. The potential for the BF class to accommodate different views on regressive transfers lifting the recipient out of poverty rests in the possibility to *unequivocally* determine the sign of Δ . The determination of a clear-cut behaviour of the index on such occasions may be considered indeed as a sensible motivation driving the choice of the value of the parameter δ , which the developers of the measure have left at the analyst’s discretion.

The satisfaction of **RA** requires $\Delta < 0$. It is straightforward to see that this condition is always met for appropriate values δ^{RA} , in fact whenever $\delta^{RA} \geq 1$ since unity is the least upper bound of $P_{\alpha>1,i}$ and consequently the ceiling to $\tilde{\Delta}$. Hence, the use of $P_{\alpha,\delta \geq 1}$ would allow a potentially very interesting behaviour: the accommodation of value judgements motivating **RA** whilst avoiding the ‘crudeness’ of $P_{\alpha=0}$.

The satisfaction of **UTA** requires $\Delta > 0$, or, equivalently, $\tilde{\Delta} > \delta$. Because of the completeness property of real numbers, this condition cannot be met by an ‘exogenous’ δ . In fact, however small we choose a δ , we can always throw from the set of real numbers an infinity of couples of numbers y_a and y_b representing the income values of donor and recipient such that $\tilde{\Delta}_{a,b} < \delta$. However, thanks to the same property of the real numbers, **UTA** can be accommodated through the choice of an ‘endogenous’ δ , i.e. a δ depending on the actual income values. In fact, once the income distribution

is given, so are the possible values of $\tilde{\Delta}$, and we can throw from the set of real numbers an infinity of δ 's smaller than the minimum value of $\tilde{\Delta}$. Consider the problem as follows.

Call I the set whose elements are the possible increments of $P_{\alpha>1}$ as a consequence of a regressive transfer lifting the recipient out of poverty - i.e. the set of possible $\tilde{\Delta}$'s - in the realm of $\hat{y} = (\hat{y}_1, \hat{y}_2, \dots, \hat{y}_n) \in \mathbb{R}_n^{>0}$, where \hat{y} is the realized income vector in society. Also, let $P(\tilde{y})$ be the poverty value associated with income \tilde{y} and let t_i indicate i 's income shortfall from the poverty line - i.e. the minimum magnitude of a transfer able to lift i out of poverty. By choosing $\delta^{UTA} < \Delta_{q-1}^+ - |\Delta_q^-| = P(y_{q-1} - t_q) - P(y_{q-1}) - P(y_q)$ it is possible to jointly accommodate **UTA** and value judgements in favour of a “fixed-plus-variable” loss from poverty motivating the BF class, because of the following general result:

PROPOSITION 1. Given a value of α , the set I has a minimum, which occurs when the donor is person $q-1$ and the recipient is person q .

For the proof of this result, see the Appendix.

One may surely question the analytical as well as the empirical attractiveness of the above condition. On the one hand, the endogenous character of δ^{UTA} makes it a less elegant result if compared with the predetermined form of δ^{RA} ; on the other hand, for populations with individuals very close to the poverty line, δ^{UTA} may happen to be very small, constraining the value the analyst can assign to the fixed loss from poverty.

Nevertheless, even in cases in which δ^{UTA} should turn out to be very small, it would at least correct the idiosyncratic values reflecting the variable loss by a common fixed loss from poverty; and, as will be illustrated below, a simple application of our new methodology shows that such constraint is not as harsh as it may appear. Furthermore, the use of an inelegant formulation might be preferred to an undesired trade-off between **UTA** and the belief in a ‘fixed-plus-variable’ loss. Finally, the spreadsheet computability of δ^{UTA} would be straightforward and when comparing income distributions A, B, \dots the use of $\min(\delta_A^{UTA}, \delta_B^{UTA}, \dots)$ will be consistent.

As a Corollary, for all δ 's such that $\delta^{UTA} < \delta \leq 1$, there exists for every potential recipient j a ‘threshold’ differentiating individuals poorer than him according to whether a ‘sacrifice’ of magnitude t_j from such a person is or is not worthwhile from a social welfare point of view. The idiosyncratic

threshold levels referred to here are evidently functions of the degree of poverty aversion exhibited by the distribution-sensitive measure. Each individual threshold would provide a ‘case by case’ identification of what Sen in the quoted passage refers to as ‘under certain circumstances’.

Finally, in this section, we apply our analytical results to the income distribution \mathbf{y} cited at the start of the paper. As may be easily verified, when $\delta \geq 1$ any regressive transfer decreasing the number of the poor leads to a decrement of the BF index for any α , according to the prescriptions of **RA**. When in the BF class $\alpha = 2$, the choice of any $\delta < 0.04$ will allow the accommodation of **UTA**, ranking distribution \mathbf{y} as having less poverty than any distribution \mathbf{y}^* obtained from \mathbf{y} through any regressive transfer. When $\alpha = 3$, any $\delta < 0.018$ will do. These may look like heavy constraints at first sight. In order to correctly appreciate their severity, however, one should consider them relative to the magnitude of the variable component of the $P_{\alpha,\delta,i}$ value – namely, the $P_{\alpha,i}$ value. Once we do that, we see that for $\alpha = 2$ the upper bound value $\delta^{UTA} = 0.04$ is the quadruple of $P_{\alpha=2,i=q}$, and for $\alpha = 3$ the bounding value $\delta^{UTA} = 0.018$ is *eighteen* times the relevant magnitude. The relative severity of such constraints decreases rapidly for larger α 's.

4. Some light on the third ‘I’, and the case for a fourth ‘I’ of poverty

In the introductory section, we called the reader’s attention to a regressive transfer turning distribution $\mathbf{y} = (\$4, \$7, \$8, \$9, \$20, \$30)$ into distribution $\mathbf{y}''' = (\$4, \$4, \$10, \$10, \$20, \$30)$. Poverty indices informative on only the incidence of poverty decrease; those informative on only the intensity of poverty remain unchanged. We asked: given that the inequality among the poor has decreased, how come that a composite measure informative on all three ‘I’s, such as $P_{\alpha=2}$, signals an increase in poverty? As a matter of fact, the introduction of the third ‘I’ into the picture reverses completely the poverty ordering – and in a direction, moreover, which is curiously opposite to the change in the third ‘I’ itself! Indeed, the third ‘I’ appears to be an irrelevant dimension of aggregate poverty if the *ceteris-paribus* assumption does not apply to both the incidence and intensity dimensions. While poverty orderings based on either of the first two ‘I’s *alone* are, though ‘crude’, justified according to some sensible views on poverty, the same cannot be said with respect to the inequality among the poor. Such an ordering would be vulnerable to well-founded criticisms, among which is, of course, the well-known “levelling-down objection”.⁴

⁴ According to this, equality obtained by making somebody worse-off and nobody better-off is a loss.

The point is not that the ordering induced by $P_{\alpha=2}$ lacks the support of sensible views on poverty, but that such views have very little to do with equality. The axioms concerning transfers among poor individuals should not be thought of as having been lifted from inequality measurement theory. The increment to the index from a regressive transfer does not derive from an *egalitarian* view, valuing equality *per se*, but stems from a *prioritarian* attitude originating in the more general principle of vertical equity, “calling for an appropriate differentiation among unequals” (Musgrave, 1990: 113). And as Broome (2007) remarks, such an attitude “leads them [prioritarians] to value equality indirectly” (p. 1).

The α -weighting of normalized income shortfalls within the sub-class $P_{\alpha>1}$ can be associated with the concept of prioritarianism on a twofold basis. Firstly, the members of this sub-class assign larger weights to lower poor incomes relative to higher poor incomes, implementing *de facto* a prioritarian weighting scheme - see Vallentyne (2003). When α is increased, the importance of worse-off individuals relative to those who are better-off grows. For example, the importance of the third worst-off individual increases relative to that of individuals 4,...,q but decreases relative to that of the worst-off and second worst-off; clearly, the worst-off individual sees his own importance increase relative to that of all other poor. Secondly, the reason which motivates such weighting scheme, when measuring absolute poverty, very closely reflects the peculiar feature of prioritarianism as expressed in the seminal work (Parfit, 1995): what matters is that “these people are at a lower *absolute* level. It is irrelevant that these people are worse off *than others* ... [but] rather that they are worse off than they might have been” (p. 23).

By means of which $\alpha \geq 1$ should we choose to prioritize worse-off individuals? Finite values of α induce forms of finitely weighted prioritarianism, whereas in the limit, as $\alpha \rightarrow \infty$, the kind of prioritarianism involved is *leximin*, which follows precisely Vallentyne’s identification of leximin weighting, in which, between two individuals, “infinitely greater weight [is given] to a worse-off person” (2003, p. 9); and within the whole population, to the worst-off person.^{5,6}

⁵ It is straightforward to show that $i < j \Rightarrow \lim_{\alpha \rightarrow \infty} (\Gamma_i)^\alpha / (\Gamma_j)^\alpha = \infty$ and $\lim_{\alpha \rightarrow \infty} (\Gamma_i)^\alpha / [(\Gamma_j)^\alpha + (\Gamma_{j+1})^\alpha + \dots + (\Gamma_q)^\alpha] = \infty$. Multiplying numerator and denominator by $1/(\Gamma_j)^\alpha$, the result becomes evident.

⁶ A fundamental difference between *leximin* and *maximin*, stressed by Vallentyne (2003), is that the latter gives absolutely no importance to the second worst-off, whereas the former requires that: 1) the situation of the worst-off should be enhanced as much as possible; 2) to the extent that the implementation of 1) allows, the situation of the second worst-off should be enhanced as much as possible, and so on with the third worst-off, fourth worst-off, etc. For P_α , we could only

Looking at the relative individual contributions to the aggregate poverty value - i.e. at the fraction $P_{\alpha,i} / P_{\alpha} = \Gamma_i^{\alpha} / [\Gamma_1^{\alpha} + \dots + \Gamma_i^{\alpha} + \dots + \Gamma_q^{\alpha}]$ - it is easy to see that the only individual whose share is always monotonically increasing in α is the worst-off, however income is distributed. When the poverty aversion parameter α grows indefinitely, only the poorest person of all tends to matter.⁷ The three I's of poverty become virtually irrelevant now, and the poverty ordering of two income distributions becomes based on a different dimension, namely the condition of the poorest. We name this dimension the **injustice** of poverty - hence a fourth 'I' of poverty - in conformity with Rawls' (1971) theory of justice entailing a special concern for the least advantaged.

Considering the injustice dimension of poverty together with the other three dimensions is of evident interest, especially in the realm of a more authentically "Rawlsian" approach to evaluation. Atkinson (1987), Vallentyne (2000) and Tungodden and Vallentyne (2006) all emphasise the misidentification, especially by economists, of the subset of society to whom Rawls (1971) addresses his difference principle.⁸ While the "least advantaged" is generally intended as strictly the worst-off individual in society, what Rawls really refers to is the least advantaged *group*. Once an appropriate cut-off function identifying the least advantaged group is set, the latter may turn out to be relatively large, and information on its condition may consequently gain a certain interest.

5. 'I's of poverty, prioritarianism and distributional properties in the FGT and BF classes

Bourguignon and Fields (1997) see their class as enjoying desirable properties related to those of $P_{\alpha>1}$ "while also combining with them the insight reflected in the headcount ratio on the loss from being poor" (p. 156). Indeed, the augmentation of $P_{\alpha>1,i}$ by a fixed loss from poverty induces an increase in the relative importance of the incidence dimension over the other dimensions of aggregate poverty.

have full maximin if we could consider comparisons - entailing either individuals or whole groups - between the poor on the one side *and the nonpoor on the other side*. However P_{α} is a focused measure. Therefore, there can be no maximin-like relationships in the concern expressed by P_{α} for poor individuals having different incomes.

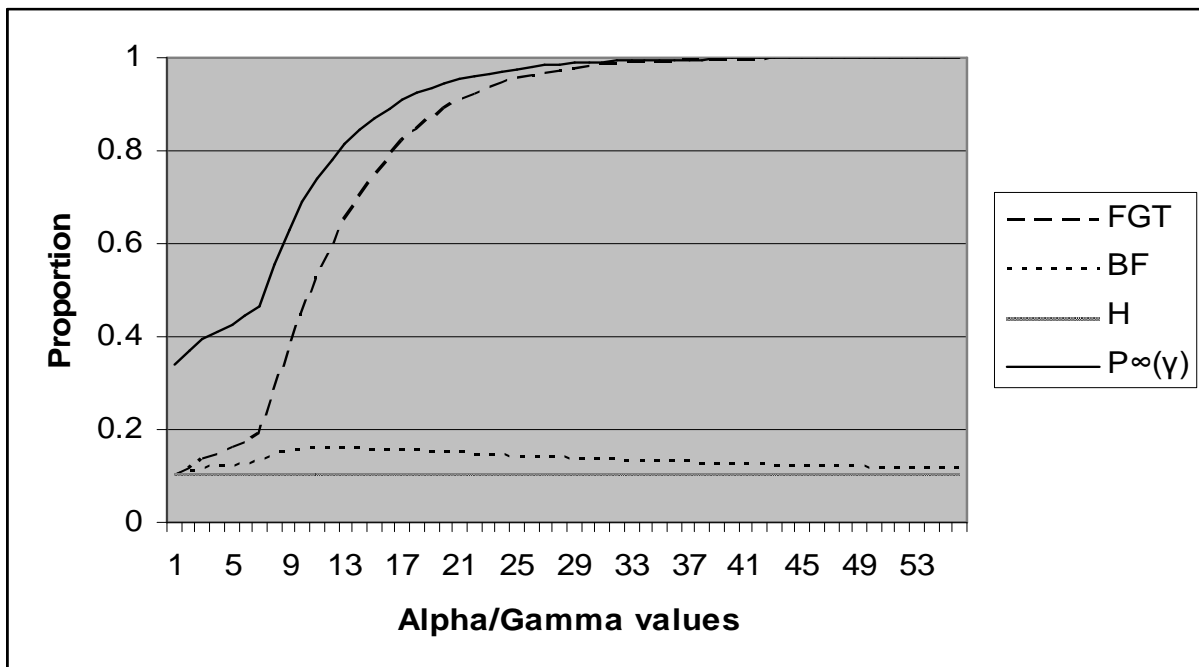
⁷ Rearranging the poorest persons's share as $P_{\alpha,i=1} / P_{\alpha} = [1 + (\Gamma_2 / \Gamma_1)^{\alpha} + (\Gamma_3 / \Gamma_1)^{\alpha} + \dots + (\Gamma_q / \Gamma_1)^{\alpha}]^{-1}$, one can see immediately that $\lim_{\alpha \rightarrow \infty} (P_{\alpha,i=1} / P_{\alpha}) = 1$.

⁸ In addition, Atkinson (1987) interestingly observes that the difference principle has nothing to do with poverty *per se*, and remarks that poverty would more naturally enter Rawls' theoretical framework through his first principle. In fact, the argument in the difference principle is an ordinal one, and the least advantaged may be well above the poverty line; instead, the first principle postulates priority to be given to the basic liberties, a necessary condition for which can be identified in a minimum income level. However, whenever the set of the poor is a non-empty set, then the difference principle is surely of interest to the extent that the least advantaged is below the poverty line.

Precisely such higher weight on the incidence of poverty is the factor enabling the corresponding BF measure $P_{\alpha,\delta}$ to depart from **UTA** in cases in which the number of poor people is affected by an income transfer.

At the individual level, the inclusion of the parameter δ delivers a mitigation of the prioritarian attitude inherent in the choice of an $\alpha > 1$. Moreover, since the $P_{\alpha,i}$ value approaches zero as $\alpha \rightarrow \infty$, in $P_{\alpha,\delta}$ only the fixed loss “survives” when α grows indefinitely. At the limit, the discriminative attitude towards different poor income levels completely vanishes and the poor individuals end up counting all equally, independently of their income levels, exactly as when $\alpha = 0$. The poorest person of all is no exception to that. As we can see from Figure 1, for $\alpha \rightarrow \infty$, while in P_α the share of the poorest in aggregate poverty approaches unity, in $P_{\alpha,\delta}$ it tends down to $1/q$ as for the headcount. It follows that for $\alpha \rightarrow \infty$, $P_{\alpha,\delta}$ has the same informational content as the headcount. Figure 1 also shows a new index, P_γ^∞ , plotted against its parameter γ , to which we shall come shortly.

Fig. 1: Contribution of the poorest to aggregate poverty : FGT, BF, H and P_γ^∞ .



As we have already explained, the choice of the poverty aversion parameter α informs the way the aggregate measure takes into account the four ‘I’s of poverty. Each member of the FGT and BF

classes with $1 < \alpha < \infty$ is informative on all four 'T's, and the choice of an α within that range may be thought of as reflecting a certain degree of 'finitely weighted prioritarianism'. This choice in turn determines the distributional properties possessed by the index. As shown in Fishburn (1980) and Fishburn and Willig (1984), successively higher transfer principles are linked with correspondingly higher moments of distributions and welfare functions - in our case, welfare-loss functions - whose derivatives alternate in sign. Once we pass from the distribution of incomes $y = (y_1, y_2, \dots, y_q)$ to that of the normalized poverty gaps $\Gamma = (\Gamma_1, \Gamma_2, \dots, \Gamma_q)$ - as we have already implicitly done in our discussion - Fishburn's results are directly applicable to the FGT and BF classes. Members of such classes are in fact built upon the power functions $P_{\alpha,i} = (\Gamma_i)^\alpha$, so that, for integer α 's, the aggregate value P_α corresponds to the α^{th} moment of the distribution of poverty gaps. Larger α 's are thus associated with higher degrees of transfer sensitivity.

PROPOSITION 2. Within the FGT and BF classes, there is no highest level of transfer sensitivity: $\forall \alpha', \exists \alpha^$ such that P_{α^*} and $P_{\alpha^*,\delta}$ accommodate a higher level of transfer sensitivity than do $P_{\alpha'}$ and $P_{\alpha',\delta}$ respectively.*

Proof. If $\alpha' \in (1, \infty)$ then $\exists \alpha^* : \alpha^* > \alpha'$, and the proof follows from Fishburn's results. If $\alpha' \in \{0, 1, \infty\}$ then the statement holds $\forall \alpha^* : 1 < \alpha^* < \infty$, since none of $P_{\alpha=0}, P_{\alpha=1}$ and $P_{\alpha \rightarrow \infty}$ are distribution sensitive; equally for $P_{\alpha \rightarrow \infty, \delta}$. *QED.*

Hence there is no upper limit to the assignation of transfer sensitivity to the FGT and BF indices. Actually, if we follow a suggestion of Zheng (1993), to quantify the welfare loss from poverty in simple percentage terms ("as... other index developers have done" (p. 84)), specifically by the ratio $\frac{z}{y_i}$, then, we see, all four 'T's of poverty are accommodated as well as the highest degrees of transfer sensitivity:

PROPOSITION 3. A poverty measure whose individual deprivation function is the percentage welfare loss from poverty $\frac{z}{y_i}$ is informative on all four 'T's of poverty and satisfies transfer sensitivity at the highest level.

Proof. $\frac{z}{y_i} = \frac{z}{y_i + z - z} = (1 - \frac{z - y_i}{z})^{-1} = \frac{1}{1 - \Gamma_i} = \Gamma_i^0 + \Gamma_i^1 + \Gamma_i^2 + \Gamma_i^3 + \dots = \lim_{M \rightarrow \infty} \sum_{\alpha=0}^M (\Gamma_i)^\alpha$. QED

6. The individual deprivation function in scale-invariant poverty indices

Define $P_i^\infty = \frac{z}{y_i}$. The significance of Proposition 3 is considerable. For as Foster and Shorrocks (1991: 701) and Zheng (1993: 85) have each pointed out, all decomposable and scale invariant poverty indices take the general form $P = a \sum_{i=1}^q \varphi(P_i^\infty)$ where, indeed, the individual loss-from-poverty contribution is a transformation of the percentage welfare loss P_i^∞ of Proposition 3.⁹ Consequently, if, for poverty measures generally the choice of the functional form of the individual deprivation function affects the informational content of the aggregate index in respect of the four ‘I’s of poverty, as well as its distributional properties, for decomposable scale-invariant measures a more precise interpretation is now available. The transformation inherent in $\varphi(\cdot)$ may be thought of as a tool, on the one hand, restraining the degree of transfer sensitivity of the index relative to the degree inherent in $a \sum_{i=1}^q \varphi(P_i^\infty)$ when $\varphi(\cdot)$ is the identity function; on the other hand, possibly decreasing the number of ‘I’s of poverty which the index takes into account, transforming the four-‘I’s-informative index into, for example, a one-‘I’-informative index such as H or the leximin poverty gap.

The effect of $\varphi(\cdot)$ on the degree of transfer sensitivity of the index has to do merely with the curvature of the individual deprivation function for $y \in (0, z)$. But, recalling the discussion in Section 3, the informational content of the index on the ‘I’s of poverty is crucially affected also by the behaviour of the individual deprivation function at the poverty line. That, in turns, informs the response of the index to transfer-led line-crossings. Rewriting P_i^∞ as $P_i^\infty = 1 + \tilde{P}_i^\infty$, one can see how P_i^∞ is a

⁹ For example, if the decomposable relative poverty index is represented by the FGT class itself, then we can write

$$P_{i,\alpha} = (1 - \frac{y_i}{z})^\alpha = \varphi(\frac{z}{y_i}), \text{ where } \varphi(\cdot) = h\left\{g\left[f(\cdot)\right]\right\} \text{ with } f(\cdot) = \left(\frac{z}{y_i}\right)^{-1}, \quad g(\cdot) = 1 - f(\cdot) \text{ and } h(\cdot) = [g(\cdot)]^\alpha. \text{ By}$$

applying to $h(\cdot)$ the function $l(\cdot) = \delta + h(\cdot)$ one obtains the BF contribution function. If $\varphi(\cdot)$ is the logarithmic function, then $P_i(y_i, z)$ becomes the individual loss-from-poverty function for the Watts (1968) poverty index,

$$P_i^W = \log z - \log y_i = \log \frac{z}{y_i}.$$

poverty-line-discontinuous, monotonic and distribution sensitive index melding fixed and variable losses from poverty precisely according to BF's prescriptions, where the fixed loss equals one and the variable loss is expressed by $\tilde{P}_i^\infty = \lim_{M \rightarrow \infty} \sum_{\alpha=1}^M (\Gamma_i)^\alpha$ - which clearly satisfies transfer sensitivity at the highest level. It follows that $\varphi(\cdot)$ is, for various indices, a transformation apt to depart from the 'fixed-plus-variable' approach in favor of unrestricted continuity and to depart from **RTA** in favor of **UTA**. For those indices, a condition such as $\varphi(1)=0$ will be verified, which means that the individual deprivation function will intersect the horizontal axis when $y_i = z$. Monotonicity turns the question of the intersection of the individual deprivation function with the vertical axis into an investigation of its upper-boundedness. Since P_i^∞ , as well as the deprivation function inherent in the Watts index, is not bounded above - which is not the case for the entire family in Hagenaars (1987), to which the FGT and BF classes themselves can be ascribed - it is evident that $\varphi(\cdot)$ may also serve to put a ceiling upon the scale-invariant index, allowing a predeterminable condition for the accommodation of **RA**.

Finally, we recall Sen's (1976) Axiom N, according to which in the case of a perfectly egalitarian distribution of incomes below the poverty line, an index fully informative on the relevant dimensions of aggregate poverty is given by HI, in his own words a "simple" and "arbitrary" multiplicative form between the indices H and I (p. 227). Following along this line a, simple and arbitrary, $\varphi(\cdot)$ taking a multiplicative form between two well-known indices may be used to take into account the relevant dimensions of poverty when poor incomes differ. Once more using the algebra of geometric series, $\varphi(P_i^\infty, \gamma) = \Gamma_i^\gamma + \Gamma_i^{\gamma+1} + \Gamma_i^{\gamma+2} + \dots = \frac{\Gamma_i^\gamma}{1 - \Gamma_i} = \frac{z}{y_i} (\Gamma_i)^\gamma = P_i^\infty P_{\gamma,i}$ is the individual deprivation function of such a decomposable index, which we name P_γ^∞ . This can be seen as a parametric generalization of P_i^∞ and \tilde{P}_i^∞ in which γ indicates both the lowest degree of transfer sensitivity and the "softest" degree of prioritarianism included in the measure. If $\gamma \geq 1$, only the variable loss from poverty is taken into account and the index will enjoy continuity-related properties; if $\gamma = 0$ the individual deprivation function is discontinuous reflecting the belief in the existence of a fixed loss from poverty alongside the variable loss. And the larger is γ , the larger is the relative weight on the fourth "T" of poverty. As can be seen in Figure 1, for correspondent values of α and γ the concern for the poorest is larger in P_γ^∞ than in P_α . Along the lines we comprehensively discussed in

this paper, this can be seen as the result of a more ‘prioritarian’ attitude of the index at the individual level, where person i ’s deprivation function for a certain γ is given by the correspondent $P_{\alpha,i}$, with the additional weight P_i^∞ being the relative welfare loss from poverty.

7. Conclusions

In the BF class, individual losses from poverty are aggregated using chosen values of the parameters α and δ . Significant consequences stem from the analyst’s choices of α and δ . As we have argued, the α -weighting at the individual level can be read in terms of prioritarianism, and the δ -value determines whether, and to what extent, a fixed loss from poverty is to be taken into account. Such choices are shown to affect the informational content of the aggregate index on Sen’s three ‘I’s of poverty. The re-examination of those ‘I’s not only led us to a discussion on how to view the inequality dimension, but also it fostered our conceptualization of a fourth ‘I’. The *injustice* of poverty, corresponding closely with Rawls’s concern for the least advantaged, should, to our eyes, be formally admitted into the picture of the poverty dimensions that are relevant for poverty evaluation.

Passing from the income distribution [vector $y = (y_1, \dots, y_q)$] to the distribution of the normalized poverty gaps [vector $\Gamma = (\Gamma_1, \dots, \Gamma_q)$], a close association is possible between the members of the BF and FGT classes for integer values of α and the moments of that latter distribution, with well-known links to transfer properties. This led to our perspective on the wide family of subgroup-decomposable scale-invariant poverty indices, not seen before, and to a new understanding of the role of the individual deprivation function in this family.

Not least, we investigated alternative value judgements behind different behaviours of a poverty index when the poverty line is crossed, and we showed how within the BF class a number of visions can be accommodated through an appropriate choice of its parameters. Ethical flexibility has been added, so that that additional possibilities for making poverty comparisons arise, bringing opportunities for deeper research and wider scope for cogent measurement.

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Appendix: the proof of Proposition 1

The increment $\Delta_D^+ - |\Delta_R^-|$ to $P_{\alpha > 1}$ following the transfer of an amount t will be minimum when the transfer exactly fills the recipient's gap from the poverty line: the transfer of any larger amount merely creates an increase in Δ_D^+ with no effect on Δ_R^- (because $P_{\alpha > 1}$ is a focused poverty measure). Hence without loss of generality we consider transfers exactly filling the recipient's shortfall from the poverty line. Proposition 1 asserts that $\Delta_{q-1}^+ - |\Delta_q^-| < \Delta_{i-h}^+ - |\Delta_i^-| \forall i, h \in \{1, 2, \dots, i-1\}$, or, equivalently, that:

$$\underbrace{P(y_{q-1} - t_q) - P(y_{q-1})}_{\Delta_D^+, D=q-1} - \underbrace{P(y_q)}_{\Delta_R^-, R=q} < \underbrace{P(y_{i-h} - t_i) - P(y_{i-h})}_{\Delta_D^+, D=i-h} - \underbrace{P(y_i)}_{\Delta_R^-, R=i}, \quad \forall i,$$

where t_q and t_i correspond to the income shortfalls from the poverty line of, respectively, the richest

poor and the i^{th} poor. Since $\frac{\partial^2 P}{\partial y_i^2} > 0$, the RHS increases with t_i and h : the more “distant” are the donor

and recipient in the RHS, the larger is the RHS. Hence it is enough to establish the inequality for the smallest possible values of t_i and h . We consequently take $h = 0$ ¹⁰ and $i = q-1$ (so that $t_i = t_{q-1}$ is the smallest t_i after t_q , which is what we have on the LHS). The inequality becomes

$$P(y_{q-1} - t_q) - P(y_{q-1}) - P(y_q) < P(y_{q-1} - t_{q-1}) - P(y_{q-1}) - P(y_{q-1}), \quad \text{or} \quad P(y_{q-1} - t_{q-1}) - P(y_{q-1} - t_q) > P(y_{q-1}) - P(y_q).$$

Now consider the differences in the arguments of $P(\cdot)$ on the LHS and RHS of this last inequality. They are the same, because $t_{q-1} = z - y_{q-1}$ and $t_q = z - y_q$. The inequality is verified

because $\frac{\partial P}{\partial y_i} < 0$ and $\frac{\partial^2 P}{\partial y_i^2} > 0$. *Q. E. D.*

¹⁰ Clearly, if $h = 0$ the transfer should not be called “regressive”. However, this choice makes it clear that the distance between the donor and the recipient in the RHS is minimized.